1. Introduction

1.1 What this Book is About

Roughly speaking, this book is about the measurement of voting power. It will take us some time, space and effort to make this rough statement — and the key concept of *voting power* — more precise; but the general idea can be outlined right away. It concerns any collective body that makes yes-or-no decisions by vote. For the sake of definiteness and brevity, we shall refer here to the decision-making body as a *board* (although it can equally well be a council, a legislature, a committee, a shareholders' meeting or a whole nation participating in a referendum); and we shall refer to a proposed resolution put to the vote as a *bill*. We shall use the term *division* to refer to the collective act whereby each board member casts a vote regarding a given bill.[1] We usually speak of the *voters* or *members* of a board as of individual persons; but as a matter of fact a group of persons may sometimes be regarded as a single member. This is the case, for example, in a legislature in which several political parties are represented: if the representatives of a party can be assumed to vote in unison, with perfect discipline, then they may collectively be regarded as a single voter, wielding a number of mandates.

Every board has a definite rule for passing bills.[2] For any such

[1] We have borrowed this term from English parliamentary usage. The literature on voting power does not always make a careful terminological distinction between the individual and the collective acts, and refers to both as 'voting'.

[2] In reality a voting body may operate several decision rules, applied to different types of resolution; see Rem. 2.1.2(iii) for the way the formal theory deals with this. For the moment, such complexities may be ignored.

rule, we may ask: to what extent is a given member able to control the outcome of a division? This 'extent' is the member's voting power. More precisely, what is to be measured is *a priori* voting power, determined without taking into consideration voters' prior bias regarding the bill voted upon, or the degree of affinity (for example, ideological proximity) between voters.[3]

The question we have just posed can be split into two parts. First, can we quantify the *absolute* voting power of a board member? Is it meaningful to compare the voting powers of members of two different boards, having two different decision rules? Second, what is the *relative share* of a given member in the decision-making power of the entire board? The distinction between absolute and relative voting power is analogous to that between income (what is the size of a given individual's income?) and income distribution (what proportion of the total income of a group accrues to each of its members?). Clearly, a positive solution to the problem of measuring absolute voting power would logically imply a solution to the problem of measuring relative voting power. The converse, however, is not true, as the following discussion demonstrates.

There are cases in which the problem of measuring relative voting power has an obvious solution: if the decision rule is *symmetric*, treating all voters in exactly the same way,[4] then all the voters have equal voting power. In such cases the relative power of each of the n voters is $1/n$. But this tells us nothing about absolute voting power.

1.1.1 Example Consider a board consisting of three members whose decision rule is that of unanimity: a bill is passed iff[5] all three members vote for it. Then the relative voting power of each member is $\frac{1}{3}$.

Now suppose that the board is enlarged by adding one new member. At the same time, the decision rule is changed to that of

---

[3]For a detailed discussion of this point see Com. 2.2.3.

[4]For a rigorous definition of 'symmetric', see Rem. 2.3.11(ii); but cf. also Def. 2.1.7 and Rem. 2.1.8.

[5]Here and throughout this book, 'iff' is short for 'if and only if'.

absolute majority: a bill is passed iff at least three (*any* three!) of the four members vote for it. Again, for reasons of symmetry, all members have equal relative power: $\frac{1}{4}$.

The *relative* voting power of each member is now smaller than before. But has the *absolute* power of the three old members changed? If so, in what way? The answer is not at all obvious. Moreover, it is not self-evident that the question itself is meaningful.[6]

There is an important class of cases where the determination of relative voting power seems simple enough, but on closer inspection this impression turns out to be deceptive. Here we have in mind *weighted voting* decision rules. Under such a rule, each board member is assigned a non-negative number as *weight*, and a certain positive number is fixed as *quota*. The rule is that a bill is passed iff the total weight of those voting for it is equal to or greater than the quota. It is tempting to jump to the conclusion that the powers of the voters are proportional to their respective weights, so that the relative power of each voter is equal to that voter's relative share of the total weight.[7] But this cannot generally be true.

1.1.2 Example Consider a board with $n+1$ members: a chairperson and n ordinary members. The chairperson has weight 1, while each ordinary member has the same weight $w > 1$. The quota is set at half the total weight: $q = (nw + 1)/2$.

First, suppose n is even, say $n = 2m$. Then a bill is passed iff at least $m + 1$ members vote for it. This rule is symmetric, because the chairperson's vote has exactly the same effect as an ordinary member's. The voting-power ratio between the chairperson and an ordinary member is $1 : 1$, while their weight ratio is $1 : w$. If w is much greater than 1, the disparity between the two ratios is very large.

Now suppose n is odd, say $n = 2m + 1$. In this case a bill is passed iff at least $m + 1$ ordinary members vote for it. The

[6] We shall return to this example in Ex. 3.2.6.

[7] This fallacy seems to have the allure of a siren: even seasoned sailors occasionally fall for it. See, for example, below § 4.2, fn. 52.

chairperson's vote makes no difference at all, so her relative voting power is 0. Clearly, all the voting power is shared equally among the n ordinary members, each having relative power $1/n$. This result holds even if w is arbitrarily close to 1.

1.1.3 Example This is a real-life example. The Council of Ministers (CM) of the European Community (EC) operates a weighted voting rule. In 1958, when the EC was first set up (as the European Economic Community), it had six members: France, [the Federal Republic of] Germany, Italy, Belgium, The Netherlands, Luxembourg. The weights in the CM were as follows: France, Germany and Italy had 4 units each; Belgium and The Netherlands had 2 units each; and Luxembourg had 1. The quota was set at 12.

It is evident that each of the three big members had more power than each of the two middle-sized ones, and the latter had more power than Luxembourg. But it would be wrong to assume that the powers were proportional to the respective weights. Luxembourg did *not* have half as much power as Belgium, or a quarter as much as Italy. In fact, Luxembourg had no voting power at all: for a bill would pass iff it were supported at least by all big three, or by two of the big three and both middle-sized members; how Luxembourg voted made absolutely no difference!

Luxembourg apart, what about the other five members, who were evidently not powerless? Would we be justified in inferring that Belgium, with weight 2, had half as much power as France, whose weight was 4? Not necessarily; for, suppose that the weights had been 8 for each of the big three, 6 for each of the middle-sized two, and 1 for Luxembourg, and the quota set at 24. Then power relations in the CM would have not changed at all: for a bill to pass, it would still need the support of all big three, or of two of the big three and both middle-sized members. Whatever the ratio between the voting powers of Belgium and France may have been, it could not possibly be both 1 : 2 and 3 : 4.

This example demonstrates that even in a comparatively simple case some effort may be required in order to determine relative voting powers, let alone absolute ones. That the effort is worth making should be clear enough: the measurement of voting power has obvious descriptive and prescriptive uses. From a descriptive viewpoint, citizens of the European Union, for example, ought surely to be interested to know how much voting power is wielded by their representatives in the CM, and in particular how the total power is shared among the various member states. From a prescriptive point of view, knowledge of this kind is vital for a rational and equitable design of decision-making rules. Indeed, as we shall see in Chapter 4, since the late 1960s this aspect of the measurement of voting power assumed legal and political importance in the United States.

As that US story illustrates, what is at issue may often be relative rather than absolute voting power.

Before leaving this section, we should like to say a few words about the relation of our topic to the main body of the theory of voting. Briefly, the measurement of voting power is, so to speak, orthogonal to the concerns of the general theory of voting, which studies the properties of various voting procedures. This is so for two reasons. First, the really interesting problems in the general theory arise in situations where a group of voters (an electorate) must choose between more than two options (candidates). On the other hand, the measurement of voting power is concerned with situations where just two outcomes are possible: a bill is either passed or rejected. Second, the general theory has occupied itself, for the most part, with symmetric decision rules (those which treat all voters in the same way). On the other hand, the study of voting power makes no such restrictions. Moreover, the literature on the subject has concentrated mainly on the problem of relative voting power, which (as we have noted) is trivial in the case of symmetric rules.[8]

---

[8]In this connection note that the problem of voting power can also be trivialized by using a probabilistic decision rule. Suppose each voter is assigned a weight, as in weighted voting; and each bill is decided by a single voter chosen at random by weighted lottery, in which the probabilities of being chosen are proportional to the members' weights. Clearly, these probabilities can be regarded as the members' respective voting powers.

1.2 Historical Sketch

In this section we draw a brief bare outline of the history of our subject. We shall list here a few landmarks, which are intended to provide chronological points of reference for the discussion in the sequel.

Many thoughtful observers must have realized that under a weighted voting rule voting power may not be proportional to weight. An early example is Luther Martin, a Maryland delegate to the 1787 Constitutional Convention held in Philadelphia. In a pamphlet published the following year he not only exposes the fallacy of equating power with weight, but makes an attempt — albeit unsystematic and somewhat crude — to measure voting power.[9]

As far as we know, the first scientific work on the measurement of voting power is Lionel Penrose's 1946 short paper [78]. The approach of this trail-blazing paper is entirely probabilistic. In particular, Penrose proposes a probabilistic measure of absolute voting power. Although this measure is defined in the context of a special class of decision rules — 'decisions made by majority vote' — it is clearly much more general, and can be applied to a very broad class of rules. Next, he presents some numerical results concerning the power of a ' "resolute" bloc' of voters 'who always vote together', while the remaining voters are 'an "indifferent" random voting group'.

He then turns to the main theme of the paper: a two-tier voting system, such as 'a federal assembly of nations' — an obvious reference to the newly established United Nations — in which a set of constituencies of different sizes elect one representative each to a decision-making 'assembly of spokesmen'. He argues that an equitable distribution of voting power in the assembly is the *square-root rule*, according to which 'the voting power of each nation in a world assembly should be proportional to the square root of the number of people on each nation's voting list'.

This paper should have been seminal; but the seed fell on stony

[9] For details see Riker [86].

ground. Penrose's ideas were subsequently re-invented by others, and his name is hardly mentioned in the mainstream literature on voting power.[10]

The mainstream's founding paper [97] was published in 1954 by Lloyd S Shapley and Martin Shubik. Their approach is essentially game-theoretic. A year earlier, the first of these authors had published his influential [94], in which he presented the *Shapley value*: a function ϕ that assigns to each cooperative game $\mathbf{v}$ and each player i in the 'universe of players' a numerical value $\phi_i(\mathbf{v})$ — the 'value' for i of playing the game $\mathbf{v}$. In [97], the authors define the *Shapley–Shubik* (briefly, S-S) *index* of voting power as a special case of the Shapley value. This is possible because a decision-making body — or, more precisely, the decision rule of such a body — can often be modelled mathematically as a particularly simple kind of cooperative game.[11] As we shall see in Com. 6.1.5, the S-S index is essentially a *relative* measure: it attributes no real meaning to absolute voting power, and the S-S values of all voters in a given game always add up to 1.[12] In [97], the authors illustrate the use of their index by applying it to several examples, including the real-life cases of the US legislature and the UN Security Council (UNSC). However, they do not specify precisely the class of cooperative games to which their index is applicable; instead, they invoke the notion of *simple game* defined by von Neumann and Morgenstern in [108]. A rigorous redefinition

[10]The exception that proves the rule is Morriss [70, p. 160], which gives him full credit, but does not belong to the mainstream. In Fielding and Liebeck [37, p. 249], which also does not belong to the mainstream, he is given credit for the square-root rule, but not for his measure of voting power. This attribution of the square-root rule is cited also in Grofman and Scarrow [41, p. 171], which does belong to the mainstream.

[11]See below, Def. 2.1.1 and Rem. 6.2.2(ii). However, this model ignores absences and abstentions by assuming that a voter who does not vote for a bill counts as voting against it. This assumption is violated by many commonly used decision rules — a fact that went largely unmentioned for a very long time (see Com. 2.2.4 and, in greater detail, § 8.1).

[12]In this book we reserve the term *index* of voting power for relative measures, whose values — as in the present case — always add up to 1.

of the class of games in question was published eight years later by Shapley [95]. His class of *simple games* is in fact considerably broader than that of [108]. In [95] the problem of measuring voting power is not addressed. However, the definitions of the notion of simple game and various auxiliary notions presented in this paper (see below, § 2.1) provided the formal mathematical infrastructure for most of the later work on voting-power theory.

The next major landmark was the publication of John F Banzhaf's paper [5] in 1965. The author, a jurist, addresses the issue of voting power from a legal-constitutional point of view: the requirement that in representative assemblies 'equal numbers of citizens have substantially equal representation' ([5, p. 317]). In the US, some state legislatures and county boards had attempted to meet this requirement by using weighted voting and assigning to board members weights proportional to the size of the population each represents. An underlying assumption was, of course, that voting power is proportional to weight. The author shows, using examples of the kind presented by us in § 1.1, that the assumption is fallacious. He then proposes a measure of voting power, based implicitly on a probabilistic approach, which is essentially the same as that proposed 19 years earlier in [78]. However, by the nature of the problem Banzhaf addresses — that of equal representation — he is only interested in determining the *ratio* of the voting power of any board member to that of any other member in the same board. So, although a measure of absolute voting power, virtually identical to that of [78], is implicit in his analysis, what he actually seems to propose is the corresponding relative measure, which became known as the *Banzhaf* (briefly, Bz) *index*.

In a second paper [6] published a year later, Banzhaf presents, essentially, a derivation of the square-root rule — unaware that in this too he is following in the footsteps of Penrose — and explores its consequences for multi-member constituencies.

The legal-constitutional issue addressed in [5] and [6] gained prominence in the US beginning in the late 1960s (see Ch. 4). This highlighted the practical importance of our topic, and no doubt helped to attract theorists to do further research into it.

In a paper [20] published in 1971, James S Coleman subjects the S-S index to a conceptual critique. He shows that the rough, informal characterization of voting power — *the extent to which a given member is able to control the outcome of a division*, as we put it in § 1.1 — can be explicated in more than one way (see below, Com. 2.2.2). He argues that the assumptions underlying the S-S index, inherited from its game-theoretic origins, make it unsuited for application to most real-life situations in which decisions are made by division. He then proposes what he regards as two new measures of voting power — quantifying the (absolute) power of members to *prevent* action and to *initiate* it, respectively — based on an alternative explication of the informal notion of voting power.

In fact, these measures were not entirely new: they were modifications of the Penrose measure, differing from it (and from each other) by a mere scaling factor. The author was apparently unaware not only of [78], as virtually everyone else was, but also of [5]. It was soon pointed out, however, that the index of relative voting power yielded by the Coleman measures is none other than the Bz index (see, for example, [14]). Thus the technical proposals of [20] could be dismissed as having no great novelty value.[13] Unfortunately, the perspicacious conceptual analysis contained in that paper was largely overlooked as well.

The S-S index and the Bz index have, by and large, been accepted as valid measures of a priori voting power. Some authors have a preference for one or another of these two indices; many regard them as equally valid. Although other indices have been proposed — notably by John Deegan and Edward Packel [22] and by R J Johnston [55], both in 1978 — none has achieved anything like general recognition as a valid index.

While the S-S index is relatively easy to handle mathematically, the Bz index is, for technical reasons, quite refractory. These technical reasons disappear if the index is rescaled by multiplying its values by an appropriate factor. In their massive 1979 paper [27], Pradeep

[13] However, the Coleman measures jointly do yield more information than the Penrose–Banzhaf measure, as we shall see in Ex. 3.2.22 and Ex. 3.2.23.

Dubey and Lloyd S Shapley propose such a rescaling as 'being in many respects more natural' than the Bz index itself ([27, p. 102]). This turns out to be essentially the Penrose measure of absolute voting power (in fact, it is precisely that measure multiplied by 2). The authors — who are of course unaware of [78] — point out the probabilistic meaning of this measure and subject it to a searching mathematical examination.

Following [27], other authors have advocated this variant of the Bz index as a measure of absolute voting power (for example, see [100, p. 267]).

From the mid-1970s, a number of investigators have pointed out various 'paradoxes' associated with the measurement of voting power. These are re-examined in [29], where it is shown that all but one of the hitherto discovered paradoxes are only paradoxical in a rather mild and superficial sense: they may be surprising to an uninformed observer but on closer examination turn out to be explicable phenomena inherent in the very notion of voting power, and are therefore displayed by any minimally reasonable method of measuring it. The remaining 'paradox' is a pathology specific to the index proposed in [22], and seems to disqualify this index as a reasonable measure of voting power.

However, two new, closely connected, severe paradoxes are demonstrated in [29], the *bloc* and *donation* paradoxes, which affect all known indices except the S-S (see § 7.8). It is argued that these new paradoxes vindicate the S-S index, at least provisionally, while the remaining indices — the most important of which is the Bz — are, at best, suspect.

This conclusion is modified in [36]. Here yet another severe paradox, the *bicameral* paradox, is shown to afflict the S-S index and all other known indices except the Bz (see § 7.9). On the other hand, the conceptual analysis of [20] is amplified, leading to a distinction between two pre-formal notions of voting power: *I-power* and *P-power*. It is argued that the concept underlying the Bz index is that of I-power, for which the bloc and donation paradoxes may be tolerable; but that as yet there exists no reasonable index of a priori P-power.